

On the Cartier Duality of Certain Finite Group Schemes of order p^n , II

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Abstract

In the previous paper [1], we have shown the Cartier duality of certain Frobenius type subgroup schemes of order p^n between Witt vectors of length n and the form of \mathbb{G}_m in positive characteristic p . In this paper, we generalize this result to the base ring having a prime number p which is a nilpotent. (However a little assumption is added.) Especially, in the proof, we give a certain endomorphism of Witt vectors which commute to Frobenius type endomorphism without assumption of characteristic p .

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1 Introduction

Throughout this paper, we denote by p a prime number. Let A be a commutative ring with unit and let λ be an arbitrary suitable element of A . We consider the deformation group scheme $\mathcal{G}^{(\lambda)}$ of the additive group scheme \mathbb{G}_a to the multiplicative group scheme \mathbb{G}_m determined by λ (we recall the group structure of $\mathcal{G}^{(\lambda)}$ in Section 3 below). The group scheme $\mathcal{G}^{(\lambda)}$ has been independently discovered by T. Sekiguchi, F. Oort and N. Suwa [7] and W. Waterhouse and B. Weisfeiler [11]. The following surjective homomorphism

$$\psi : \mathcal{G}^{(\lambda)} \rightarrow \mathcal{G}^{(\lambda^p)}; x \mapsto \lambda^{-p} \{ (1 + \lambda x)^p - 1 \}$$

is useful in the unified Kummer-Artin-Schreier theory. Remark that ψ is nothing but Frobenius homomorphism over the base ring of characteristic p .

Y. Tsuno [10] have shown the following:

Theorem 1 ([10]) *Assume that A is an \mathbb{F}_p -algebra. Then the Cartier dual of $\text{Ker}(\psi)$ is canonically isomorphic to $\text{Ker}[F - \lambda^{p-1} : \mathbb{G}_{a,A} \rightarrow \mathbb{G}_{a,A}]$, where F is Frobenius endomorphism.*

Tsuno's result is a special case of the result obtained by F. Oort and J. Tate [6]. But an important viewpoint is what embedding the certain classified finite group schemes of order p into $\mathcal{G}^{(\lambda)}$ over $A[\sqrt[p-1]{b}]$, as $\lambda = \sqrt[p-1]{b}$ for an element b in A .

The author have generalized Tsuno's theorem as follows. Let l be a positive integer. The following surjective homomorphism

$$\psi^{(l)} : \mathcal{G}^{(\lambda)} \rightarrow \mathcal{G}^{(\lambda^{p^l})}; \quad x \mapsto \lambda^{-p^l} \{(1 + \lambda x)^{p^l} - 1\},$$

implies $\psi^{(l)}(x) = x^{p^l}$ since A is an \mathbb{F}_p -algebra. Put $N_l := \text{Ker}(\psi^{(l)})$. Suppose W_A is the Witt ring scheme over A . Let $F : W_A \rightarrow W_A$ be the Frobenius endomorphism and let $[\lambda] : W_A \rightarrow W_A$ be the Teichmüller lifting of $\lambda \in A$. Put $F^{(\lambda)} := F - [\lambda^{p-1}]$. Restrict $F^{(\lambda)}$ to the Witt ring scheme $W_{l,A}$ of length l . The result of previous paper [1] is the following:

Theorem 2 ([1]) *Assume that A is an \mathbb{F}_p -algebra. Then the Cartier dual of N_l is canonically isomorphic to $\text{Ker}[F^{(\lambda)} : W_{l,A} \rightarrow W_{l,A}]$.*

T. Sekiguchi and N. Suwa [9] have introduced the deformed Artin-Hasse exponential series. Theorem 2 have been proved by using these formal power series.

This paper generalize Theorem 2, i.e., no assumption of characteristic p . Our arguments are as follows. Let $\mathbb{Z}_{(p)}$ be a localization of rational integers \mathbb{Z} at p . We suppose that A is a $\mathbb{Z}_{(p)}/(p^l)$ -algebra of locally noetherian of dimension at most 1 for the representability. Then A is $\mathbb{Z}_{(p)}$ -algebra and $p^l = 0$ in A . Let λ be an element of A . Assume that λ^{p^l} divide $p^{l-k}\lambda^{p^k}$ for any integer $0 \leq k \leq l$. Then the homomorphism $\psi^{(l)}$ becomes well-defined under this assumption and $N_l = \text{Ker}(\psi^{(l)})$ is the finite group scheme of order p^l since the degree of monic polynomial $\psi^{(l)}(X)$ is p^l . For each vector \mathbf{a} in $W(A)$, T. Sekiguchi and N. Suwa [8] have introduced endomorphism $T_{\mathbf{a}}$ on $W(A)$ (we recall the definition of $T_{\mathbf{a}}$ in Section 2 below). Take a vector $\mathbf{a} := \lambda^{-p^l} p^l [\lambda]$ in $W(A)$. Put $W(A)/T_{\mathbf{a}} := \text{Coker}[T_{\mathbf{a}} : W(A) \rightarrow W(A)]$ and $T'_{\mathbf{a}} := F^{(\lambda)} \circ T_{\mathbf{a}}$. We consider the following diagram:

$$\begin{array}{ccc} W(A) & \longrightarrow & W(A)/T_{\mathbf{a}} \\ F^{(\lambda)} \downarrow & & \downarrow \overline{F^{(\lambda)}} \\ W(A) & \longrightarrow & W(A)/T'_{\mathbf{a}}. \end{array}$$

Here the homomorphism $\overline{F^{(\lambda)}}$ is defined by $\overline{F^{(\lambda)}}(\overline{\mathbf{x}}) := \overline{F^{(\lambda)}(\mathbf{x})}$. It is checked immediately that $\overline{F^{(\lambda)}}$ is the well-defined homomorphism. Then the result of this paper is obtained as follows:

Theorem 3 *With the above notations, the Cartier dual of N_l is canonically isomorphic to $\text{Ker}[\overline{F^{(\lambda)}} : W_A/T_{\mathbf{a}} \rightarrow W_A/T'_{\mathbf{a}}]$.*

In the end of Section 4 we treat on the representability of the quotient functor $W_A/T_{\mathbf{a}}$. The framework of the proof is similar to the previous paper [1]. But we do not assume characteristic p . Therefore, Frobenius endomorphism F and Verschiebung endomorphism V are not commutative. In our arguments, it is important to give the endomorphism $T'_{\mathbf{a}}$ satisfying the condition $F^{(\lambda)} \circ T_{\mathbf{a}} = T'_{\mathbf{a}} \circ F^{(\lambda^{p^l})}$.

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Notation

- $\mathbb{G}_{a,A}$: additive group scheme over A
- $\mathbb{G}_{m,A}$: multiplicative group scheme over A
- $\widehat{\mathbb{G}}_{m,A}$: multiplicative formal group scheme over A
- $W_{n,A}$: group scheme of Witt vectors of length n over A
- W_A : group scheme of Witt vectors over A
- F : Frobenius endomorphism of W_A
- $[\lambda]$: Teichmüller lifting $(\lambda, 0, 0, \dots) \in W(A)$ of $\lambda \in A$
- $F^{(\lambda)}$: $= F - [\lambda^{p-1}]$
- $T_{\mathbf{a}}$: homomorphism decided by $\mathbf{a} \in W(A)$ (recalled in Section 2)
- $\mathbb{U}^{(p)}$: $= (U_0^p, U_1^p, \dots)$ for a vector $\mathbb{U} = (U_0, U_1, \dots)$
- $\frac{1}{\Lambda}\mathbb{U}$: $= (\frac{U_0}{\Lambda}, \frac{U_1}{\Lambda}, \dots)$ for a vector $\mathbb{U} = (U_0, U_1, \dots)$
- $W(A)^{F^{(\lambda)}}$: $= \text{Ker}[F^{(\lambda)} : W(A) \rightarrow W(A)]$
- $W(A)/F^{(\lambda)}$: $= \text{Coker}[F^{(\lambda)} : W(A) \rightarrow W(A)]$
- $W(A)/T_{\mathbf{a}}$: $= \text{Coker}[T_{\mathbf{a}} : W(A) \rightarrow W(A)]$

2 Witt vectors

In this Section we recall necessary facts on Witt vectors and their homomorphisms for this paper. For details, see [3, Chap. V] or [5, Chap. III].

2.1

Let $\mathbb{X} = (X_0, X_1, \dots)$ be a sequence of variables. For each $n \geq 0$, we denote by $\Phi_n(\mathbb{X}) = \Phi_n(X_0, X_1, \dots, X_n)$ the Witt polynomial

$$\Phi_n(\mathbb{X}) = X_0^{p^n} + pX_1^{p^{n-1}} + \dots + p^n X_n$$

in $\mathbb{Z}[\mathbb{X}] = \mathbb{Z}[X_0, X_1, \dots]$. Let $W_{n,\mathbb{Z}} = \text{Spec}(\mathbb{Z}[X_0, X_1, \dots, X_{n-1}])$ be an n -dimensional affine space. The phantom map $\Phi^{(n)}$ is defined by

$$\Phi^{(n)} : W_{n,\mathbb{Z}} \rightarrow \mathbb{A}_{\mathbb{Z}}^n; \mathbf{x} \mapsto (\Phi_0(\mathbf{x}), \Phi_1(\mathbf{x}), \dots, \Phi_{n-1}(\mathbf{x})),$$

where $\mathbb{A}_{\mathbb{Z}}^n$ is the usual n -dimensional affine space over \mathbb{Z} . The scheme $\mathbb{A}_{\mathbb{Z}}^n$ has a natural ring scheme structure. It is known that $W_{n,\mathbb{Z}}$ has a unique commutative ring scheme structure over \mathbb{Z} such that the phantom map $\Phi^{(n)}$ is a homomorphism over \mathbb{Z} . Then the point of $W_{n,\mathbb{Z}}$ is called Witt vector of length n over \mathbb{Z} .

2.2

The endomorphism $F : W(A) \rightarrow W(A)$ is defined by

$$\Phi_i(F(\mathbf{x})) = \Phi_{i+1}(\mathbf{x})$$

for $\mathbf{x} \in W(A)$. If A is an \mathbb{F}_p -algebra, F is nothing but the usual Frobenius endomorphism. For $\lambda \in A$, we denote by $[\lambda]$ the Teichmüller lifting $[\lambda] = (\lambda, 0, 0, \dots) \in W(A)$. We put the endomorphism $F^{(\lambda)} := F - [\lambda^{p-1}]$ on $W(A)$.

For each $\mathbf{a} = (a_0, a_1, \dots) \in W(A)$, the additive endomorphism $T_{\mathbf{a}} : W(A) \rightarrow W(A)$ is defined by

$$\Phi_n(T_{\mathbf{a}}(\mathbf{x})) = a_0^{p^n} \Phi_n(\mathbf{x}) + pa_1^{p^{n-1}} \Phi_{n-1}(\mathbf{x}) + \dots + p^n a_n \Phi_0(\mathbf{x})$$

for $\mathbf{x} \in W(A)$ ([8, Chap.4, p.20]).

3 Deformed Artin-Hasse exponential series

In this Section we recall necessary facts on the deformed Artin-Hasse exponential series for this paper. For details, see [9] and [8].

3.1

Let A be a ring and let λ be an element of A . Put $\mathcal{G}^{(\lambda)} := \text{Spec}(A[X, 1/(1 + \lambda X)])$. We define a morphism $\alpha^{(\lambda)}$ by

$$\alpha^{(\lambda)} : \mathcal{G}^{(\lambda)} \rightarrow \mathbb{G}_{m,A}; x \mapsto 1 + \lambda x.$$

It is known that $\mathcal{G}^{(\lambda)}$ has a unique group scheme structure over A such that the morphism $\alpha^{(\lambda)}$ is a homomorphism over A . Then the group scheme structure of $\mathcal{G}^{(\lambda)}$ is given by $x \cdot y = x + y + \lambda xy$. If λ is invertible in A , $\alpha^{(\lambda)}$ is an A -isomorphism. On the other hand, if $\lambda = 0$, $\mathcal{G}^{(\lambda)}$ is nothing but the additive group scheme $\mathbb{G}_{a,A}$.

3.2

The Artin-Hasse exponential series $E_p(X)$ is given by

$$E_p(X) = \exp \left(\sum_{r \geq 0} \frac{X^{p^r}}{p^r} \right) \in \mathbb{Z}_{(p)}[[X]].$$

The formal power series $E_p(U, \Lambda; X) \in \mathbb{Q}[U, \Lambda][[X]]$ is defined by

$$E_p(U, \Lambda; X) = (1 + \Lambda X)^{\frac{U}{\Lambda}} \prod_{k=1}^{\infty} (1 + \Lambda^{p^k} X^{p^k})^{\frac{1}{p^k} \left((\frac{U}{\Lambda})^{p^k} - (\frac{U}{\Lambda})^{p^{k-1}} \right)}.$$

As in [9, Corollary 2.5.] or [8, Lemma 4.8.], it has shown that this formal power series $E_p(U, \Lambda; X)$ is in $\mathbb{Z}_{(p)}[U][[X]]$. Note that $E_p(1, 0; X) = E_p(X)$.

Suppose A be a $\mathbb{Z}_{(p)}$ -algebra. Let λ be an element of A and let $\mathbf{v} = (v_0, v_1, \dots)$ be a vector in $W(A)$. The formal power series $E_p(\mathbf{v}, \lambda; X)$ is defined by

$$E_p(\mathbf{v}, \lambda; X) = \prod_{k=0}^{\infty} E_p(v_k, \lambda^{p^k}; X^{p^k}) = (1 + \lambda X)^{\frac{v_0}{\lambda}} \prod_{k=1}^{\infty} (1 + \lambda^{p^k} X^{p^k})^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(F^{(\lambda)}(\mathbf{v}))}. \quad (1)$$

Moreover the formal power series $F_p(\mathbf{v}, \lambda; X, Y)$ is defined as follows:

$$F_p(\mathbf{v}, \lambda; X, Y) = \prod_{k=1}^{\infty} \left(\frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k} (X + Y + \lambda XY)^{p^k}} \right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(F^{(\lambda)}(\mathbf{v}))}. \quad (2)$$

As in [9, Lemma 2.16.] or [8, Lemma 4.9.], it has shown that the formal power series $F_p(\mathbf{v}, \lambda; X, Y)$ is in $A[[X, Y]]$. The formal power series $F_p(F^{(\lambda)}\mathbf{v}, \lambda; X, Y)$ has the following equalities:

$$\begin{aligned} F_p(F^{(\lambda)}\mathbf{v}, \lambda; X, Y) &= \prod_{k=1}^{\infty} \left(\frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k} (X + Y + \lambda XY)^{p^k}} \right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(F^{(\lambda)}(\mathbf{v}))} \\ &= \frac{E_p(\mathbf{v}, \lambda; X) E_p(\mathbf{v}, \lambda; Y)}{E_p(\mathbf{v}, \lambda; X + Y + \lambda XY)}. \end{aligned} \quad (3)$$

T. Sekiguchi and N. Suwa [9] have shown the following isomorphisms with the defined above formal power series (1) and (2):

$$W(A)^{F^{(\lambda)}} \xrightarrow{\sim} \text{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}); \quad \mathbf{v} \mapsto E_p(\mathbf{v}, \lambda; X), \quad (4)$$

$$W(A)/F^{(\lambda)} \xrightarrow{\sim} H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}); \quad \mathbf{w} \mapsto F_p(\mathbf{w}, \lambda; X, Y) \quad (5)$$

([9, Theorem 2.19.1.]). Here $H_0^2(G, H)$ denotes the Hochschild cohomology group consisting of symmetric 2-cocycles of G with coefficients in H for formal group schemes G and H ([3, Chap. II.3 and Chap. III.6]).

Put vectors $\mathbb{U} := (U_0, U_1, \dots)$ and $\mathbb{V} := \frac{1}{\Lambda_2} \mathbb{U} = (\frac{U_0}{\Lambda_2}, \frac{U_1}{\Lambda_2}, \dots)$. Set the following:

$$E := E_p(\mathbb{U}, \Lambda_1; X) \quad \text{and} \quad E^{(p^r)} := E_p(\mathbb{U}^{(p^r)}, \Lambda_1^{p^r}; X^{p^r}).$$

For a vector $\mathbb{W} := (W_0, W_1, \dots)$, the following formal power series is defined:

$$\tilde{E}_p(\mathbb{W}, \Lambda_2; E) := E^{\frac{W_0}{\Lambda_2}} \prod_{r=1}^{\infty} (E^{(p^r)})^{\frac{1}{p^r \Lambda_2^{p^r}} \Phi_{r-1}(F^{(\Lambda_2)}(\mathbb{W}))}.$$

As in [8, Proposition 4.11.], it has shown that the following equality:

$$\tilde{E}_p(\mathbb{W}, \Lambda_2; E) = E_p(T_{\mathbb{V}}(\mathbb{W}), \Lambda_1; X). \quad (6)$$

Put the following:

$$G_p(\mathbb{W}, \Lambda_2; E) := \prod_{l \geq 1} \left(\frac{1 + (E - 1)^{p^l}}{E^{(p^l)}} \right)^{\frac{1}{p^l \Lambda_2^{p^l}} \Phi_{l-1}(\mathbb{W})}. \quad (7)$$

Then the following equality holds:

$$G_p(F^{(\Lambda_2)}\mathbb{W}, \Lambda_2; E) = \frac{E_p(\mathbb{W}, \Lambda_2; \frac{1}{\Lambda_2}(E - 1))}{\tilde{E}_p(\mathbb{W}, \Lambda_2; E)}. \quad (8)$$

By the equalities (6), (7) and (8) of formal power series, the following equality holds:

$$E_p(T_{\mathbb{V}}(\mathbb{W}), \Lambda_1; X) = E_p(\mathbb{W}, \Lambda_2; \frac{1}{\Lambda_2}(E - 1)) \cdot G_p(F^{(\Lambda_2)}\mathbb{W}, \Lambda_2; E). \quad (9)$$

4 Proof of Theorem 3

A technical portion of the proof is prepared in Subsection 4.1. In Subsection 4.2 we complete our proof of Theorem 3.

4.1

In Subsection 4.1, we show that the homomorphism $T'_a := F^{(\lambda)} \circ T_a$ have the condition $F^{(\lambda)} \circ T_a = T'_a \circ F^{(\lambda^{p^l})}$, especially.

Suppose that A is a ring. Let λ be an element of A and let l be a positive integer. Assume that λ^{p^l} divide $p^{l-k}\lambda^{p^k}$ for any integer $0 \leq k \leq l$. Put a vector $\mathbf{a} := \lambda^{-p^l} p^l[\lambda] \in W(A)$.

Lemma 1 *With the above notations, the following equality holds:*

$$\text{Ker}(F^{(\lambda)} \circ T_a) = \text{Ker}(F^{(\lambda^{p^l})}).$$

Proof First, we calculate the components of $\mathbf{b} := p^l[\lambda] \in W(A)$ by using the phantom map $\Phi^{(n)}$. For $\mathbf{b} = (b_0, b_1, \dots)$, we have $b_0 = p^l\lambda$ by $\Phi_0(\mathbf{b}) = \Phi_0(p^l[\lambda])$. Similarly, we have $b_1 = p^{l-1}\lambda^p(1 - p^{(p-1)l})$. Put $\alpha_1 := (1 - p^{(p-1)l})$. For $k \geq 2$, we can find the following:

$$b_k = p^{l-k}\lambda^{p^k} (1 - p^{(p^{k-1}-1)l} - p^{(p^{k-1}-1)(l-1)}\alpha_1^{p^{k-1}} - p^{(p^{k-2}-1)(l-2)}\alpha_2^{p^{k-2}} - \dots - p^{p-1}\alpha_{k-1}^p)$$

inductively, where

$$\alpha_k := 1 - p^{(p^k-1)l} - \sum_{i=1}^{k-1} p^{(p^k-i-1)(l-i)} \alpha_i^{p^{k-i}} \quad (k \geq 2). \quad (10)$$

Remark that

$$b_k \equiv \lambda^{p^l} \pmod{p} \text{ if } k = l \quad \text{and} \quad b_k \equiv 0 \pmod{p} \text{ if } k \neq l. \quad (11)$$

Therefore the vector \mathbf{b} is stated the following explicitly:

$$\mathbf{b} = p^l[\lambda] = (p^l\lambda, p^{l-1}\lambda^p\alpha_1, p^{l-2}\lambda^{p^2}\alpha_2, \dots, \lambda^{p^l}\alpha_l, p^{-1}\lambda^{p^{l+1}}\alpha_{l+1}, \dots). \quad (12)$$

Moreover we also obtain the components of $\mathbf{a} = \lambda^{-p^l}\mathbf{b} \in W(A)$.

Next, we show the equality of Lemma 1. For $\mathbf{x} \in \text{Ker } F^{(\lambda^{p^l})}$, we have $\Phi_{k+1}(\mathbf{x}) = \lambda^{p^{l+k}(p-1)}\Phi_k(\mathbf{x})$ by the phantom map since $F^{(\lambda^{p^l})}(\mathbf{x}) = F(\mathbf{x}) - [\lambda^{p^l(p-1)}] \cdot \mathbf{x} = \mathbf{o}$. In particular, we have $\Phi_1(\mathbf{x}) = \lambda^{p^l(p-1)}\Phi_0(\mathbf{x})$. Then we claim $F^{(\lambda)} \circ T_a(\mathbf{x}) = \mathbf{o}$. Put $\mathbf{y} := F^{(\lambda)} \circ T_a(\mathbf{x})$. For $\mathbf{y} = (y_0, y_1, y_2, \dots)$, we have

$$y_0 = \Phi_0(\mathbf{y}) = \Phi_0(F \circ T_a(\mathbf{x})) - \lambda^{p-1}\Phi_0(T_a(\mathbf{x})) = (a_0^p\lambda^{p^l(p-1)} + pa_1 - \lambda^{p-1}a_0)\Phi_0(\mathbf{x}).$$

By using the relation (10) of the components of \mathbf{a} , we have $\lambda^{p^l(p-1)}a_0^p + pa_1 - \lambda^{p-1}a_0 = 0$. Hence we have $y_0 = 0$. The claim is shown by the induction. By using the phantom map, the following equalities are hold:

$$\begin{aligned} \Phi_k(F^{(\lambda)} \circ T_a(\mathbf{x})) &= \Phi_{k+1}(T_a(\mathbf{x})) - \lambda^{p^k(p-1)}\lambda^{p^{k-1}(p-1)} \dots \lambda^{p-1}\Phi_0(T_a(\mathbf{x})) \\ &= \lambda^{p^{p+k}(p-1)}\lambda^{p^{p+k-1}(p-1)} \dots \lambda^{p^l(p-1)}a_0^{p^{k+1}}\Phi_0(\mathbf{x}) + \dots \\ &\quad + \lambda^{p^{p+k-1}(p-1)}\lambda^{p^{p+k-2}(p-1)} \dots \lambda^{p^l(p-1)}pa_1^{p^k}\Phi_0(\mathbf{x}) + \dots + p^{k+1}a_{k+1}\Phi_0(\mathbf{x}) - \lambda^{p^{k+1}-1}a_0\Phi_0(\mathbf{x}) \\ &= (\lambda^{p^{l+k+1}-p^l}a_0^{p^{k+1}} + p\lambda^{p^{l+k}-p^l}a_1^{p^k} + \dots + p^{k+1}a_{k+1} - \lambda^{p^{k+1}-1}a_0)\Phi_0(\mathbf{x}) \\ &= \{p^{k+1}a_{k+1} - \frac{p^l\lambda^{k+1}}{\lambda^{p^l}}(1 - p^{(p^{k+1}-1)l} - p^{(p^k-1)(l-k)}\alpha_1^{p^k} - \dots - p^k \cdot p^{p(l-k)}\lambda^{p^{k+1}}\alpha_k^p)\}\Phi_0(\mathbf{x}) \\ &= \frac{p^l\lambda^{p^{k+1}}}{\lambda^{p^l}}\{\alpha_{k+1} - (1 - p^{(p^{k+1}-1)l} - \sum_{i=1}^k p^{(p^{k+1}-i)(l-i)}\alpha_i^{p^{k+1-i}})\}\Phi_0(\mathbf{x}) = 0. \end{aligned}$$

Hence we have $\Phi'_k(\mathbf{x}) = 0$ for any integer k . Therefore $F^{(\lambda)} \circ T_a(\mathbf{x}) = \mathbf{o}$ holds for any \mathbf{x} in $\text{Ker}(F^{(\lambda)})$. In order to show the claim $\text{Ker}(F^{(\lambda^{p^l})}) \supset \text{Ker}(F^{(\lambda)} \circ T_a)$, we consider the

following diagram:

$$\begin{array}{ccc}
W(A) & \xrightarrow{T_a} & W(A) \\
\Delta \downarrow & & \downarrow F^{(\lambda)} \\
W(A) \times W(A) & & \\
(F, -[\lambda^{p^l(p-1)}]) \downarrow & & \\
W(A) \times W(A) & \xrightarrow{t'_a} & W(A) \times W(A) \\
m \downarrow & & \searrow m \\
W(A) & \xrightarrow{T'_a} & W(A),
\end{array}$$

where

$$m : W(A) \times W(A) \rightarrow W(A); (\mathbf{x}_1, \mathbf{x}_2) \mapsto \mathbf{x}_1 + \mathbf{x}_2$$

and

$$\Delta : W(A) \rightarrow W(A) \times W(A); \mathbf{x} \mapsto (\mathbf{x}, \mathbf{x}).$$

Put $\mathbf{c} := \lambda^{-p^{l+1}} p^l [\lambda]$. We define the homomorphism

$$\begin{aligned}
t'_a : W(A) \times W(A) &\rightarrow W(A) \times W(A); \\
(\mathbf{x}_1, \mathbf{x}_2) &\mapsto (T_{a(p)}(\mathbf{x}_1), T_{c(p)} \circ F(\mathbf{x}_2) - F \circ T_c(\mathbf{x}_2) + [\lambda^{p-1}] \circ T_c(\mathbf{x}_2)).
\end{aligned}$$

The homomorphism t'_a is well-defined over $(\text{Im}(F)) \times (\text{Im}(-[\lambda^{p^l(p-1)}]))$. Therefore we obtain the following equalities:

$$F^{(\lambda)} \circ T_a = m \circ t'_a \circ (F, -[\lambda^{p^l(p-1)}]) \circ \Delta \quad \text{and} \quad F^{(\lambda^{p^l})} = m \circ (F, -[\lambda^{p^l(p-1)}]) \circ \Delta.$$

Hence we obtain the result by $m^{-1}(\mathbf{o}) \supset t_a^{-1} \circ m^{-1}(\mathbf{o})$. □

By Lemma 1, we have the isomorphisms

$$\text{Im}(F^{(\lambda^{p^l})}) \xleftarrow{\sim} W(A)/\text{Ker}(F^{(\lambda^{p^l})}) = W(A)/\text{Ker}(F^{(\lambda)} T_a) \xrightarrow{\sim} \text{Im}(F^{(\lambda)} T_a).$$

Therefore, by putting $T'_a := F^{(\lambda)} \circ T_a$, the equality $F^{(\lambda)} \circ T_a = T'_a \circ F^{(\lambda^{p^l})}$ holds by the identification of $\text{Im}(F^{(\lambda^{p^l})}) \simeq W(A)/\text{Ker}(F^{(\lambda^{p^l})})$, i.e., we identify the following way:

$$\begin{aligned}
\alpha : \text{Im}(F^{(\lambda^{p^l})}) \cup W(A) \setminus \text{Im}(F^{(\lambda^{p^l})}) &\xrightarrow{\sim} W(A)/\text{Ker}(F^{(\lambda^{p^l})}) \cup W(A) \setminus \text{Im}(F^{(\lambda^{p^l})}); \\
\alpha(\mathbf{y}) &:= \overline{\mathbf{x}} \quad \text{if } \mathbf{y} = F^{(\lambda^{p^l})}(\mathbf{x}) \in \text{Im}(F^{(\lambda^{p^l})}) \quad \text{and} \quad \alpha(\mathbf{x}) := \mathbf{x} \quad \text{if } \mathbf{x} \in W(A) \setminus \text{Im}(F^{(\lambda^{p^l})}).
\end{aligned}$$

The group structure is also induced to $W(A)/\text{Ker}(F^{(\lambda^{p^l})}) \cup W(A) \setminus \text{Im}(F^{(\lambda^{p^l})})$ by α .

4.2

In this Subsection, we complete our proof of Theorem 3.

In addition to the assumption of Subsection 4.1, we assume that A is a $\mathbb{Z}_{(p)}/(p^l)$ -algebra of locally noetherian of dimension at most 1 for the representability. Then we can take an element $\lambda \in A$ as $\lambda = 0$ or invertible element, obviously. At least, in addition to this, we can take λ as nilpotent element as follows. Put $\zeta := e^{\frac{2\pi i}{p^l}}$. Set $A := \mathbb{Z}_{(p)}[\zeta]/(p^l)$. Take as $\lambda = 1 - \zeta \in A$. Then, by the following equality:

$$X^{p^l-1} + X^{p^l-2} + \cdots + X + 1 = (X - \zeta)(X - \zeta^2) \cdots (X - \zeta^{p^l-1}),$$

we have $(p) = (\lambda^{p^l-1}) \subset A$ as $X = 1$. Therefore λ^{p^l} divide $p^l \lambda$. Moreover, λ^{p^l} can also divides $p^{l-k} \lambda^{p^k}$ since ζ^{p^k} divides ζ^{p^l} for any integer $0 \leq k \leq l$. Then λ is a nilpotent element.

Let $\mathcal{G}^{(\lambda)}$ be the deformation group scheme defined in Subsection 3.1 and $\widehat{\mathcal{G}}^{(\lambda)}$ be the formal completion of $\mathcal{G}^{(\lambda)}$ along the zero section. We consider the following homomorphism:

$$\psi^{(l)} : \widehat{\mathcal{G}}^{(\lambda)} \rightarrow \widehat{\mathcal{G}}^{(\lambda^{p^l})}; x \mapsto \lambda^{-p^l} \{(1 + \lambda x)^{p^l} - 1\}.$$

Here the following equalities hold:

$$\psi^{(l)}(X) = \lambda^{-p^l} \{(1 + \lambda X)^{p^l} - 1\} = \lambda^{-p^l} \sum_{k=1}^{p^l-1} \binom{p^l}{k} \lambda^k X^k + X^{p^l}.$$

Hence we can write $\overline{X}^{p^l} = p\overline{R}$. By the our assumption which p is a nilpotent, we have the kernel of the homomorphism $\psi^{(l)}$ has the following equalities:

$$N_l := \text{Ker } \psi^{(l)} = \text{Spf } A[[X]]/(\psi^{(l)}(X)) = \text{Spec } A[X]/(\psi^{(l)}(X)).$$

Remark that the formal scheme N_l is nothing but the finite group scheme of order p^l , since the class \overline{X} is nilpotent and the degree of $\psi^{(l)}(X)$ is p^l . The homomorphism $\psi^{(l)}$ induce the following short exact sequence (13):

$$0 \longrightarrow N_l \xrightarrow{\iota} \widehat{\mathcal{G}}^{(\lambda)} \xrightarrow{\psi^{(l)}} \widehat{\mathcal{G}}^{(\lambda^{p^l})} \longrightarrow 0,$$

where ι is a canonical inclusion. This exact sequence deduces the following long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \text{Hom}(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) \xrightarrow{(\psi^{(l)})^*} \text{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \xrightarrow{(\iota)^*} \text{Hom}(N_l, \widehat{\mathbb{G}}_{m,A}) \\ &\xrightarrow{\partial} \text{Ext}^1(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) \xrightarrow{(\psi^{(l)})^*} \text{Ext}^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \longrightarrow \cdots \end{aligned}$$

Since the images of the boundary map ∂ and the map $(\psi^{(l)})^*$ are given by direct products of schemes we can replace $\text{Ext}^1(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A})$ and $\text{Ext}^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A})$ with $H_0^2(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A})$

and $H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A})$, respectively ([1, Lemma 3 and Lemma 4]). Therefore we have the following exact sequence (14):

$$\begin{array}{ccccc} \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) & \xrightarrow{(\psi^{(l)})^*} & \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) & \xrightarrow{(\iota)^*} & \mathrm{Hom}(N_l, \widehat{\mathbb{G}}_{m,A}) \\ & \xrightarrow{\partial} & H_0^2(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) & \xrightarrow{(\psi^{(l)})^*} & H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}). \end{array}$$

We consider the following diagram (15):

$$\begin{array}{ccccccc} W(A) & \xrightarrow{T_a} & W(A) & \xrightarrow{\pi} & W(A)/T_a & \longrightarrow & 0 \\ F^{(\lambda^{p^l})} \downarrow & & F^{(\lambda)} \downarrow & & \overline{F^{(\lambda)}} \downarrow & & \\ W(A) & \xrightarrow{T'_a} & W(A) & \longrightarrow & W(A)/T'_a & \longrightarrow & 0. \end{array}$$

The exactness of the horizontal sequences are obvious. The commutativity of the squares already have checked in Introduction and Subsection 4.1. To apply the snake lemma, we define the boundary map ∂ as follows. Put $M_l := \mathrm{Ker}[\overline{F^{(\lambda)}} : W(A)/T_a \rightarrow W(A)/T'_a]$. The boundary map ∂ becomes the canonical map by the definition of T'_a . The map ∂ is well-defined and a group homomorphism. Moreover the exactness is also obviously. Therefore we put $\partial(\overline{w}) := \overline{w}$ for each $\overline{w} \in M_l$. Thus we can apply the snake lemma for the diagram (15). Therefore we have the following exact sequence (16):

$$\begin{array}{ccccccc} W(A)^{F^{(\lambda^{p^l})}} & \xrightarrow{T_a} & W(A)^{F^{(\lambda)}} & \xrightarrow{\pi} & M_l & & \\ & \xrightarrow{\partial} & W(A)/F^{(\lambda^{p^l})} & \xrightarrow{V^l} & W(A)/F^{(\lambda)}. & & \end{array}$$

Now, by combining the exact sequences (14), (16) and the isomorphisms (4), (5), we have the following diagram (17) consisting of exact horizontal lines and vertical isomorphisms except ϕ :

$$\begin{array}{ccccccc} \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) & \xrightarrow{(\psi^{(l)})^*} & \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) & \xrightarrow{(\iota)^*} & \mathrm{Hom}(N_l, \widehat{\mathbb{G}}_{m,A}) & & \\ \phi_1 \uparrow & & \phi_2 \uparrow & & \phi \uparrow & & \\ W(A)^{F^{(\lambda^{p^l})}} & \xrightarrow{T_a} & W(A)^{F^{(\lambda)}} & \xrightarrow{\pi} & M_l & & \\ & & & & & & \\ & \xrightarrow{\partial} & H_0^2(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) & \xrightarrow{(\psi^{(l)})^*} & H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) & & \\ & & \phi_3 \uparrow & & \phi_4 \uparrow & & \\ & \xrightarrow{\partial} & W(A)/F^{(\lambda^{p^l})} & \xrightarrow{T'_a} & W(A)/F^{(\lambda)}, & & \end{array}$$

where ϕ is the following homomorphism induced from the exact sequence (13) and the isomorphism (4):

$$\phi : M_l \rightarrow \mathrm{Hom}(N_l, \widehat{\mathbb{G}}_{m,A}); \quad \overline{x} \mapsto E_p(\mathbf{x}, \lambda; x).$$

We must check the well-definedness of the homomorphism ϕ . Take an element $\overline{\mathbf{x}}$ of M_l . Then, by $\mathbf{x} + T_a(\mathbf{x}_0) \in W(A)^{F^{(\lambda)}}$, the following congruences are hold:

$$\begin{aligned} E_p(\overline{\mathbf{x}}, \lambda; x) &= E_p(\mathbf{x}, \lambda; x) \cdot E_p(T_a(\mathbf{x}_0), \lambda; x) \equiv E_p(\mathbf{x}, \lambda; x) \cdot E_p(\mathbf{x}_0, \lambda; \psi^{(l)}(x)) \pmod{p^l} \\ &\equiv E_p(\mathbf{x}, \lambda; x) \pmod{\psi^{(l)}(x)}. \end{aligned}$$

Here we need the following lemma:

Lemma 2 *Under the above notations, the following congruence holds:*

$$E_p(\mathbf{x}, \lambda^{p^l}; \psi^{(l)}(x)) \equiv E_p(T_a(\mathbf{x}), \lambda; x) \pmod{p^l}.$$

Proof Take $\mathbf{x} \in W(A)$. By the equality (9), we have the following equalities:

$$\begin{aligned} E_p(\mathbf{x}, \lambda^{p^l}; \psi^{(l)}(x)) &= E_p(\mathbf{x}, \lambda^{p^l}; \frac{1}{\lambda^{p^l}} \{E_p(p^l[\lambda], \lambda; x)^{p^l} - 1\}) \\ &= \widetilde{E_p}(\mathbf{x}, \lambda^{p^l}; E_p(p^l[\lambda], \lambda; x)) \cdot G_p(F^{(\lambda^{p^l})}(\mathbf{x}), \lambda^{p^l}; E_p(p^l[\lambda], \lambda; x)) \\ &= E_p(T_a(\mathbf{x}), \lambda; x) \quad \text{if } G_p \equiv 1 \pmod{p^l}. \end{aligned}$$

Therefore we must show the following congruence:

$$G_p(F^{(\lambda^{p^l})}(\mathbf{x}), \lambda^{p^l}; E_p(p^l[\lambda], \lambda; x)) \equiv 1 \pmod{p^l}.$$

Here, by the equality (7), we have the following:

$$G_p(F^{(\lambda^{p^l})}(\mathbf{x}), \lambda^{p^l}; E_p(p^l[\lambda], \lambda; x)) = \prod_{k \geq 1} \left(\frac{1 + (E_p(p^l[\lambda], \lambda; x) - 1)^{p^k}}{E_p(p^l[\lambda], \lambda; x)^{(p^k)}} \right)^{\frac{1}{p^k \Lambda p^k} \Phi_{k-1}(F^{(\lambda)} \circ T_a(\mathbf{x}))}.$$

Since $E_p(p^l[\lambda], \lambda; x) = (1 + \lambda x)^{p^l}$ we obtain the following congruence:

$$1 + (E_p(p^l[\lambda], \lambda; x) - 1)^{p^k} \equiv 1 + \lambda^{p^{l+k}} x^{p^{l+k}} \pmod{p^{k+1}}.$$

On the other hand, we must show the following congruence:

$$E_p^{(p^k)}(p^l[\lambda], \lambda; x) \equiv 1 + \lambda^{p^{l+k}} x^{p^{l+k}} \pmod{p^{k+1}}.$$

By expanding on $\mathbf{b} = p^l[\lambda]$, we have the following congruence:

$$E_p(\mathbf{b}, \lambda; x) \equiv 1 + \sum_{r \geq 0} b_r x^{p^r} \pmod{(\lambda^{p^i} b_i, b_i b_j)}.$$

Therefore, by congruence (11), we obtain the following result

$$E_p(\mathbf{b}^{(p^k)}, \lambda^{p^k}; x^{p^k}) \equiv 1 + \sum_{r \geq 0} b_r^{p^k} x^{p^{k+r}} \pmod{(\lambda^{p^{k+i}} b_i^{p^k}, b_i^{p^k} b_j^{p^k})}.$$

Remark that $\Phi_{k-1}(F^{(\lambda)} \circ T_a(\mathbf{x}))$ is divided by p^l , at least. Hence the result. \square

If the commutativity of the diagram (17) is checked, the five lemma shows that ϕ is isomorphism, i.e., $M_l \simeq \text{Hom}(N_l, \widehat{\mathbb{G}}_{m,A})$. To check of the diagram (17), we show the following congruences under modulo p^l :

$$\begin{aligned} (1) : (\psi^{(l)})^* \circ \phi_1 &\equiv \phi_2 \circ T_a, & (2) : (\iota)^* \circ \phi_2 &\equiv \phi \circ \pi, \\ (3) : \partial \circ \phi &\equiv \phi_3 \circ \partial, & (4) : (\psi^{(l)})^* \circ \phi_3 &\equiv \phi_4 \circ T'_a. \end{aligned}$$

For the congruences (1) and (2), we must show the following:

$$E_p(\mathbf{x}, \lambda^{p^l}; \psi^{(l)}(x)) \equiv E_p(T_a(\mathbf{x}), \lambda; x) \pmod{p^l}.$$

But we already have proved the above congruence. For the congruences (3) and (4), we must show the following:

$$F_p(\mathbf{x}, \lambda^{p^l}; \psi^{(l)}(x), \psi^{(l)}(y)) \equiv F_p(T'_a(\mathbf{x}), \lambda; x, y) \pmod{p^l}.$$

Remark that the calculations of the boundary ∂ and $(\psi^{(l)})^*$ are completely same to previous paper [1, Lemma 3 and Lemma 4]. But, by the equality (3), we must show the following congruence:

$$\begin{aligned} F_p(T'_a(\mathbf{x}), \lambda; x, y) &= F_p(F^{(\lambda)} \circ T_a(\mathbf{x}), \lambda;) \\ &= \frac{E_p(T_a(\mathbf{x}), \lambda; x) E_p(T_a(\mathbf{x}), \lambda; y)}{E_p(T_a(\mathbf{x}), \lambda; x + y + \lambda xy)}. \end{aligned}$$

By Lemma 2, this congruence already have proved.

Here we must check the representability of the functor W_A/T_a . We remark that $W_A/\text{Ker}(T_a)$ is representable ([2, Theorem 4.C]). Therefore, by $\text{Im}(T_a) \simeq W_A/\text{Ker}(T_a)$, $\text{Im}(T_a)$ is also representable. On the Affiness, the nilpotency of p is an obstruction. By using the base change to the reduced ring A' , we can use the Chevalley's theorem [4, III, Ex 4.2]. Therefore Theorem 3 was obtained.

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